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Explicitly B-preinvex functions ☆

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Abstract

This paper introduces a new class of functions, to be referred to as explicitly B-preinvex functions. Some properties of explicitly B-preinvex functions are established, e.g., any local minimum of an explicitly B-preinvex function is also a global one and the summation of two functions, which are both B-preinvex and explicitly B-preinvex, is also a B-preinvex function and an explicitly B-preinvex function. Furthermore, it is shown that the explicit B-preinvexity, together with the intermediate-point B-preinvexity, implies B-preinvexity, while the explicit B-preinvexity, together with a lower semicontinuity, implies the B-preinvexity. © 2002 Published by Elsevier Science B.V.

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1. Introduction

Bector and Singh [1] introduced a class of functions, which are called B-vex functions. These are generalizations of convex functions. They established that various properties, which hold for convex functions, are also valid for B-vex functions. Furthermore, some generalizations of B-vex functions have been given in [2,5]. Specifically, Suneja et al. [5] introduced the concept of B-preinvex functions which includes the B-vex as a special case. It is clear that a B-vex function coincides with a quasiconvex function.

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Mohan and Neogy [4] introduced the following Condition C: Let $\eta:\mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$; we say that the function η satisfies the Condition C if, for any x, y and for any $\lambda \in [0,1]$

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y),$$

$$\eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y),$$

and applied it to show that a differentiable invex function with respect to η is also preinvex. Mohan and Neogy also gave an example which shows that Condition C may hold for a general class of functions η, rather than just for the case $\eta(x, y) = x - y$.

It is known [3] that f is invex on an invex set K if and only if every stationary point is a global minimum of f over K and that a local minimum is also a global one for a preinvex function over an invex set. Recently, Suneja et al. [5] also obtained that a local minimum of a B-preinvex function over an invex set is also a global one. This property is very important in mathematical programming.

In this paper, we study the generalization of B-vex and B-preinvex functions and introduce a new class of functions called explicitly B-preinvex functions. It is worth noting that B-preinvexity and explicit B-preinvexity are two different properties of a function. But we show that a local minimum of an explicitly B-preinvex function over an invex set is also a global one, which is similar to that of a B-preinvex function, and that there are still some relationships between B-preinvex and explicitly B-preinvex functions. For example, using Condition C, we show that the explicit B-preinvexity, together with an intermediate-point B-preinvexity property, implies the B-preinvexity, while the explicit B-preinvexity, together with a lower semicontinuity, implies the B-preinvexity. So it is very interesting to study properties of explicitly B-preinvex functions.

The outline of the paper is as follows. In Section 2, we introduce the concept of the explicit B-preinvexity. In Section 3, we give some properties of explicitly B-preinvex functions that are obtained by using the Condition C. For example, we show that the summation of two functions, which are both B-preinvex and explicitly B-preinvex, is also a B-preinvex function and an explicitly B-preinvex function. In Section 4, several sufficient conditions of the B-preinvexity are given using the explicit B-preinvexity and the Condition C.

2. Explicitly *B*-preinvex functions

A set $K \subseteq \mathfrak{R}^n$ is said to be invex if there exists a vector function $\eta:\mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ such that

$$x, y \in K, \quad \lambda \in [0, 1] \;\Rightarrow\; y + \lambda\eta(x, y) \in K.$$

Suneja et al. [5] introduced B-preinvex functions as follows.

Definition 2.1 (Suneja et al. [5]). Let K be a nonempty subset of $\mathfrak{R}^n$ which is invex at $y \in X$ with respect to $\eta:\mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$. The function $f:K \to \mathfrak{R}$ is said to be B-preinvex at $y \in X$ with respect to η, b_1, b_2 if

$$f(y + \lambda\eta(x, y)) \leqslant b_1(x, y, \lambda)f(y) + b_2(x, y, \lambda)f(x) \quad \forall x, y \in K,\ \lambda \in [0, 1], \tag{2.1}$$

where

$$b_1(x,y,\lambda) \geqslant 0, \quad b_2(x,y,\lambda) \geqslant 0, \quad b_1(x,y,\lambda)+b_2(x,y,\lambda)=1, \quad b_1(x,y,0)=1=b_2(x,y,1).$$

The function f is said to be B-preinvex on X with respect to η, b_1, b_2 if f is B-preinvex at each $y \in X$ with respect to η, b_1, b_2; and f is said to be strictly B-preinvex on X with respect to η, b_1, b_2 if a strict inequality holds in (2.1) for any $x, y \in X$, $x \neq y$.

We now introduce a new class of generalized convex functions to be referred to as explicitly B-preinvex functions.

Definition 2.2. Let $K \subseteq \mathfrak{R}^n$ be an invex set with respect to $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$. Let $f : K \to \mathfrak{R}$. We say that f is an explicitly B-preinvex with respect to η, b_1, b_2 if $\forall x, y \in K$, $f(x) \neq f(y)$, we have

$$f(y+\lambda\eta(x,y)) < b_1(x,y,\lambda)f(y) + b_2(x,y,\lambda)f(x), \quad \forall \lambda \in (0,1),$$

where

$$b_1(x,y,\lambda) > 0, \quad b_2(x,y,\lambda) > 0, \quad b_1(x,y,\lambda)+b_2(x,y,\lambda)=1, \quad b_1(x,y,0)=1=b_2(x,y,1).$$

Remark 2.1. If $\eta(x,y) = x - y$, $b_2(x,y,\lambda) = \lambda$, $b_1(x,y,\lambda) = 1 - \lambda$, then an explicitly B-preinvex function with respect to η, b_1, b_2 is reduced to an explicitly convex function in [6].

The following examples illustrate that B-preinvexity does not imply explicitly B-preinvexity and explicitly B-preinvexity does not imply B-preinvexity.

Example 2.1. This example illustrates that a B-preinvex function is not necessarily an explicitly B-explicitly function. Let

$$f(x) = -|x|,$$

$$\eta(x,y) = \begin{cases} x-y & \text{if } x \geqslant 0, y \geqslant 0, \\ x-y & \text{if } x \leqslant 0, y \leqslant 0, \\ y-x & \text{if } x \leqslant 0, y \geqslant 0, \\ y-x & \text{if } x \geqslant 0, y \leqslant 0. \end{cases}$$

Then, f is a B-preinvex function with respect to η on $\mathfrak{R}$, where $b_1(x,y,\lambda) = 1-\lambda$, $b_2(x,y,\lambda) = \lambda$. However, by letting $y=1$, $x=2$, $\lambda = \frac{1}{2}$, we have $f(y) = f(1) = -1 > -2 = f(2) = f(x)$ and

$$f(y+\lambda\eta(x,y)) = f(1+\tfrac{1}{2}\eta(2,1)) = f(3/2) = -3/2 = b_1(x,y,\lambda)f(y) + b_2(x,y,\lambda)f(x).$$

Thus, f is not an explicitly B-preinvex function with respect to η on $\mathfrak{R}$.

Example 2.2. This example illustrates that an explicitly B-preinvex function is not necessarily a B-preinvex function. Let

$$f(x) = \begin{cases} -|x| & \text{if } |x| \leqslant 1, \\ -1 & \text{if } |x| \geqslant 1, \end{cases}$$

$$\eta(x,y) = \begin{cases} x-y & \text{if } x \geqslant 0, y \geqslant 0, \\ x-y & \text{if } x \leqslant 0, y \leqslant 0, \\ x-y & \text{if } y < -1, x > 1, \\ x-y & \text{if } x < -1, y > 1, \\ y-x & \text{if } -1 \leqslant x \leqslant 0, y \geqslant 0, \\ y-x & \text{if } -1 \leqslant y \leqslant 0, x \geqslant 0, \\ y-x & \text{if } 0 \leqslant x \leqslant 1, y \leqslant 0, \\ y-x & \text{if } 0 \leqslant y \leqslant 1, x \leqslant 0. \end{cases}$$

Then, f is an explicitly B-preinvex function with respect to η on $\mathfrak{R}$, where $b_1(x,y,\lambda) = 1-\lambda$, $b_2(x,y,\lambda) = \lambda$. However, by letting $x = 2$, $y = -2$, $\lambda = \frac{1}{2}$, we have

$$\begin{aligned} f(y + \lambda\eta(x,y)) &= f(-2 + \tfrac{1}{2}\eta(2,-2)) = f(0) = 0 > -1 \\ &= f(2) = f(-2) = b_1(x,y,\lambda)f(y) + b_2(x,y,\lambda)f(x). \end{aligned}$$

That is, f is not a B-preinvex function with respect to the same η.

3. Properties of explicitly B-preinvex functions

In this section, we derive some properties of explicitly B-preinvex functions. In particular, Theorem 3.1 below shows that a local minimum of an explicitly B-preinvex function over an invex set is also a global one.

Theorem 3.1. *Let K be a nonempty invex set in $\mathfrak{R}^n$ with respect to $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$, and $f : K \to \mathfrak{R}$ an explicitly B-preinvex function with respect to η, b_1, b_2. If $\bar{x} \in K$ is a local minimum to the problem of minimizing $f(x)$ subject to $x \in K$, then $\bar{x}$ is a global one.*

Proof. Suppose that $\bar{x} \in K$ is a local minimum to the problem of minimizing $f(x)$ subject to $x \in K$. Then there is an ε-neighborhood $N_\varepsilon(\bar{x})$ around $\bar{x}$ such that

$$f(\bar{x}) \leqslant f(x), \quad \forall x \in K \cap N_\varepsilon(\bar{x}). \tag{3.1}$$

If $\bar{x}$ is not a global minimum of f over K, then there exists an $x^* \in K$ such that

$$f(x^*) < f(\bar{x}).$$

By the explicit B-preinvexity of f with respect to η, b_1, b_2,

$$f(\bar{x} + \alpha\eta(x^*, \bar{x})) < b_2(x, y, \lambda)f(x^*) + b_1(x, y, \lambda)f(\bar{x}) < f(\bar{x})$$

for all $0 < \alpha < 1$. For a sufficiently small $\alpha > 0$, it follows that

$$\bar{x} + \alpha\eta(x^*, \bar{x}) \in K \cap N_\varepsilon(\bar{x}),$$

which is a contradiction to (3.1). This completes the proof. $\square$

By Theorem 3.1, we can conclude that explicitly B-preinvex functions constitute an important class of generalized convex functions in mathematical programming. The function Example 2.1 is not explicitly B-preinvex with respect to any η based on Theorem 3.1.

Next, we derive some properties of explicitly B-preinvex functions.

Theorem 3.2. *Let K be a nonempty invex set in $\mathfrak{R}^n$ with respect to $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n, f : K \to \mathfrak{R}$ an explicitly B-preinvex function on K with respect to η, b_1, b_2, and $g : I \to \mathfrak{R}$ a convex and strictly increasing function, where range$(f) \subseteq I$. Then the composite function $g(f)$ is an explicitly B-preinvex function on K with respect to the same η, b_1, b_2.*

Proof. For any $x, y \in K$, $\lambda \in (0, 1)$, if $g(f(x)) \neq g(f(y))$, then $f(x) \neq f(y)$. Since f is an explicitly B-preinvex function, we have

$$f(y + \lambda\eta(x, y)) < b_2(x, y, \lambda)f(x) + b_1(x, y, \lambda)f(y).$$

From convexity and strictly increasing property of g as well as $b_1 + b_2 = 1$, we obtain

$$\begin{aligned} g[f(y + \lambda\eta(x, y))] &< g[b_2(x, y, \lambda)f(x) + b_1(x, y, \lambda)f(y)] \\ &\leqslant b_2(x, y, \lambda)g(f(x)) + b_1(x, y, \lambda)g(f(y)). \end{aligned}$$

Hence, $g(f)$ is an explicitly B-preinvex function on K. $\square$

Similarly, we can prove the following two results.

Theorem 3.3. *Let K be a nonempty invex set in $\mathfrak{R}^n$ with respect to $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$, $f : K \to \mathfrak{R}$ an explicitly B-preinvex function with respect to η, b_1, b_2, and $g : I \to \mathfrak{R}$ a strictly convex and increasing function, where range$(f) \subseteq I$. Then the composite function $g(f)$ is an explicitly B-preinvex function on K with respect to the same η, b_1, b_2.*

Theorem 3.4. *Let K be a nonempty invex set in $\mathfrak{R}^n$ with respect to $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$, and $f_i : K \to \mathfrak{R}$, $i = 1, \ldots, p$, both B-preinvex functions and explicitly B-preinvex functions on K with respect to η, b_1, b_2. Then*

$$f = \sum_{i=1}^{p} \lambda_i f_i, \quad \forall \lambda_i > 0, \ \ i = 1, 2, \ldots, p$$

is both a B-preinvex function and an explicitly B-preinvex function on K with respect to the same η, b_1, b_2.

Before deriving further properties of explicitly B-preinvex functions, we present a property of B-preinvex functions.

Theorem 3.5. *Let* $K \subseteq \mathfrak{R}^n$ *be an invex set with respect to* $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$, *and* $(f_i)_{i\in I}$ *a family of real-valued functions which are B-preinvex on K with respect to* η, b_1, b_2 *and bounded from above on K, then function* $f(x) = \sup_{i\in I} f_i(x)$ *is a B-preinvex function on K with respect to the same* η, b_1, b_2.

Proof. Since each f_i $(i\in I)$ is a B-preinvex function with respect to the same function η on K, we have for each $i\in I$

$$f_i(y + \lambda\eta(x, y)) \leqslant b_1(x, y, \lambda) f_i(y) + b_2(x, y, \lambda) f_i(x), \quad \forall x, y\in K, \ \ \lambda\in[0, 1].$$

Therefore, for each $i\in I$,

$$f_i(y + \lambda\eta(x, y)) \leqslant b_1(x, y, \lambda) \sup_{i\in I} f_i(y) + b_2(x, y, \lambda) \sup_{i\in I} f_i(x), \quad \forall x, y\in K, \ \ \lambda\in[0, 1].$$

Taking sup of the left-hand side of the above equation, we obtain

$$\sup_{i\in I} f_i(y + \lambda\eta(x, y)) \leqslant b_1(x, y, \lambda) \sup_{i\in I} f_i(y) + b_2(x, y, \lambda) \sup_{i\in I} f_i(x), \quad \forall x, y\in K, \ \ \lambda\in[0, 1].$$

That is, f is a B-preinvex function on K with respect to η, b_1, b_2. $\square$

We do not know whether Theorem 3.5 still holds if the B-preinvexity is replaced by the explicit B-preinvexity. But we have the following result:

Theorem 3.6. *Let K be a nonempty invex set in* $\mathfrak{R}^n$ *with respect to* $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ *where* η *satisfies the Condition* C, *and* $\{f_i\colon i\in I\}$ *be a finite or infinite collection of both explicitly B-preinvex functions and B-preinvex functions on K with respect to the same* η, b_1, b_2. *Define*

$$f(x) = \sup\{f_i(x),\ i\in I\}, \quad x\in K.$$

Assume that for every $x\in K$, *there exists* $i_0 := i(x)\in I$, *such that* $f(x) = f_{i_0}(x)$. *Then f is both an explicitly B-preinvex function and a B-preinvex function on K with respect to* η, b_1, b_2.

Proof. By Theorem 3.5, we know that f is a B-preinvex function on K. It suffices to show that f is an explicitly B-preinvex function on K. Assume that f is not an explicitly B-preinvex function. Then, there exist $x, y\in K$, $f(x) \neq f(y)$ and $\alpha\in(0, 1)$ such that

$$f(y + \alpha\eta(x, y)) \geqslant b_2(x, y, \alpha) f(x) + b_1(x, y, \alpha) f(y).$$

By the B-preinvexity of f with respect to η, b_1, b_2, we have

$$f(y+\alpha\eta(x,y)) \leqslant b_2(x,y,\alpha)f(x)+b_1(x,y,\alpha)f(y).$$

Hence

$$f(y+\alpha\eta(x,y)) = b_2(x,y,\alpha)f(x)+b_1(x,y,\alpha)f(y). \tag{3.2}$$

Let $z=y+\alpha\eta(x,y)$. From the assumptions of the theorem, there exist $i(z):=i_0$, $i(x):=i_1$, $i(y):=i_2$, satisfying

$$f(z)=f_{i_0}(z), \quad f(x)=f_{i_1}(x), \quad f(y)=f_{i_2}(y).$$

Then, (3.2) implies that

$$f_{i_0}(z)=b_2(x,y,\alpha)f_{i_1}(x)+b_1(x,y,\alpha)f_{i_2}(y). \tag{3.3}$$

(i) If $f_{i_0}(x)\neq f_{i_0}(y)$, then by explicit B-preinvexity of f_{i_0} with respect to η, b_1, b_2, we have

$$f_{i_0}(z) < b_2(x,y,\alpha)f_{i_0}(x)+b_1(x,y,\alpha)f_{i_0}(y). \tag{3.4}$$

From $f_{i_0}(x)\leqslant f_{i_1}(x)$, $f_{i_0}(y)\leqslant f_{i_2}(y)$, and (4), we obtain

$$f_{i_0}(z) < b_2(x,y,\alpha)f_{i_1}(x)+b_1(x,y,\alpha)f_{i_2}(y),$$

which contradicts (3.3).

(ii) If $f_{i_0}(x)=f_{i_0}(y)$, then by B-preinvexity of f_{i_0} with respect to η, b_1, b_2, we have

$$f_{i_0}(z)\leqslant b_2(x,y,\alpha)f_{i_0}(x)+b_1(x,y,\alpha)f_{i_0}(y)=f_{i_0}(x)=f_{i_0}(y). \tag{3.5}$$

Since $f(x)\neq f(y)$, at least one of the inequalities $f_{i_0}(x)\leqslant f_{i_1}(x)=f(x)$ and $f_{i_0}(y)\leqslant f_{i_2}(y)=f(y)$ has to be a strict inequality. From (3.5), we obtain

$$f(z)=f_{i_0}(z) < b_2(x,y,\alpha)f(x)+b_1(x,y,\alpha)f(y),$$

which contradicts (3.3). This completes the proof. □

Theorem 3.7. *Suppose that f and $-g$ are both B-preinvex and explicitly B-preinvex on an invex set $K\subseteq\mathfrak{R}^n$ with respect to η, b_1, b_2. Furthermore, suppose that $f(x)\geqslant 0$, $g(x)>0$ for all $x\in K$. Then, $h=f/g$ is both B-preinvex and explicitly B-preinvex on K with respect to η and some $\bar{b}_1, \bar{b}_2$.*

Proof. For $x,u\in K$, $0\leqslant\lambda\leqslant 1$, from $h(x)\neq h(y)$, we know that $f(x)\neq f(u)$ or that $g(x)\neq g(u)$. By the hypothesis of f and g, we have

$$\begin{aligned} h(u+\lambda\eta(x,u)) &= f(u+\lambda\eta(x,u))/g(u+\lambda\eta(x,u)) \\ &< (b_1f(u)+b_2f(x))/(b_1g(u)+b_2g(x)) \\ &= \bar{b}_1h(u)+\bar{b}_2h(x), \end{aligned}$$

where

$$\bar{b}_1 = b_1 g(u)/(b_1 g(u) + b_2 g(x)),$$

$$\bar{b}_2 = b_2 g(x)/(b_1 g(u) + b_2 g(x)), \quad \bar{b}_1 + \bar{b}_2 = 1. \qquad \square$$

Remark. If the condition that f and $-g$ are B-preinvex on an invex set $K \subseteq \mathfrak{R}^n$ with respect to η, b_1, b_2 in Theorem 3.8 is removed, then the result of Theorem 3.7 does not hold. But we have following result.

Theorem 3.8. *Suppose that $-g$ is explicitly B-preinvex on an invex set $K \subseteq \mathfrak{R}^n$ with respect to η, b_1, b_2. Furthermore, suppose that $g(x) > 0$ for all $x \in K$. Then, $h = 1/g$ is explicitly B-preinvex on K with respect to η and some $\bar{b}_1, \bar{b}_2$.*

Proof. For $x, u \in K$, $0 \leqslant \lambda \leqslant 1$, from $h(x) \neq h(y)$, we know that $g(x) \neq g(u)$. By the explicit B-preinvexity of $-g$, we have

$$\begin{aligned} h(u + \lambda\eta(x,u)) &= 1/g(u + \lambda\eta(x,u)) \\ &< 1/(b_1 g(u) + b_2 g(x)) \\ &= \overline{b_1} h(u) + \overline{b_2} h(x), \end{aligned}$$

where

$$\bar{b}_1 = b_1 g(u)/(b_1 g(u) + b_2 g(x)),$$

$$\bar{b}_2 = b_2 g(x)/(b_1 g(u) + b_2 g(x)), \quad \bar{b}_1 + \bar{b}_2 = 1.$$

Thus h is explicitly B-preinvex on K. $\square$

4. Sufficient conditions of B-preinvexity

In this section we obtain two sufficient conditions for the B-preinvexity.

Theorem 4.1. *Let K be a nonempty invex set in $\mathfrak{R}^n$ with respect to $\eta : \mathfrak{R}^n \times \mathfrak{R}^n \to \mathfrak{R}^n$ where η satisfies the Condition C, and $f : K \to \mathfrak{R}$ an explicitly B-preinvex function on K with respect to η, b_1, b_2. If there exists $\alpha \in (0,1)$ such that for every $x, y \in K$ the following inequality holds*:

$$f(y + \alpha\eta(x,y)) \leqslant b_1(x,y,\alpha) f(y) + b_2(x,y,\alpha) f(x), \tag{4.1}$$

then f is a B-preinvex function on K with respect to η, b_1, b_2.

Proof. By contradiction, suppose that there exist $x, y \in K$ and $\lambda \in (0,1)$ such that

$$f(y+\lambda\eta(x,y)) > b_1(x,y,\lambda)f(y) + b_2(x,y,\lambda)f(x).$$

Note that λ cannot be equal to α from the assumed condition in (4.1). Without loss of generality, assume that $f(x) \geqslant f(y)$ and let

$$z = y + \lambda\eta(x,y).$$

Then

$$f(z) > b_1(x,y,\lambda)f(y) + b_2(x,y,\lambda)f(x). \tag{4.2}$$

If $f(x) > f(y)$, since f is an explicitly B-preinvex function with respect to η, b_1, b_2, we have

$$f(z) < b_1(x,y,\lambda)f(y) + b_2(x,y,\lambda)f(x),$$

which is a contradiction to (4.2).

If $f(x) = f(y)$, then (4.2) implies that

$$f(z) > f(x) = f(y). \tag{4.3}$$

(i) If $0 < \lambda < \alpha < 1$, let

$$z_1 = y + \frac{\lambda}{\alpha}\eta(x,y).$$

Thus, from Condition C

$$\begin{aligned} y + \alpha\eta(z_1, y) &= y + \lambda\eta\left(y + \frac{\lambda}{\alpha}\eta(x,y), y + \frac{\lambda}{\alpha}\eta(x,y) - \frac{\lambda}{\alpha}\eta(x,y)\right) \\ &= y + \lambda\eta\left(y + \frac{\lambda}{\alpha}\eta(x,y), y + \frac{\lambda}{\alpha}\eta(x,y) + \eta\left(y, y + \frac{\lambda}{\alpha}\eta(x,y)\right)\right) \\ &= y - \lambda\eta\left(y, y + \frac{\lambda}{\alpha}\eta(x,y)\right) \\ &= y + \lambda\eta(x,y) = z. \end{aligned}$$

According to (4.1), we have

$$f(z) \leqslant b_1(y,z_1,\alpha)f(y) + b_2(y,z_1,\alpha)f(z_1).$$

Because of (4.3) and the above inequality, it follows that

$$f(z) < f(z_1). \tag{4.4}$$

Let

$$d = \frac{(1-\alpha)\lambda}{\alpha(1-\lambda)}.$$

Because of $0<\lambda<\alpha<1$, it is easy to show that $0<d<1$. Thus, from Condition C,

$$\begin{aligned} z+d\eta(x,z) &= y+\lambda\eta(x,y)+d\eta(x,y+\lambda\eta(x,y)) \\ &= y+[\lambda+d(1-\lambda)]\eta(x,y) = y+\left[\lambda+\frac{1-\alpha}{\alpha}\lambda\right]\eta(x,y) \\ &= y+\frac{\lambda}{\alpha}\eta(x,y)=z_1. \end{aligned}$$

Since f is an explicitly B-preinvex function, from inequality (4.3) and above equality, we obtain

$$f(z_1)<b_1(x,z,d)f(z)+b_2(x,z,d)f(x)<f(z),$$

which contradicts (4.4).

(ii) If $0<\alpha<\lambda<1$, that is,

$$0<\frac{\lambda-\alpha}{1-\alpha}<1.$$

Let

$$z_2=y+\frac{\lambda-\alpha}{1-\alpha}\eta(x,y).$$

Thus, from Condition C,

$$\begin{aligned} z_2+\alpha\eta(x,z_2) &= y+\frac{\lambda-\alpha}{1-\alpha}\eta(x,y)+\alpha\eta\left(x,y+\frac{\lambda-\alpha}{1-\alpha}\eta(x,y)\right) \\ &= y+\left[\frac{\lambda-\alpha}{1-\alpha}+\alpha\left(1-\frac{\lambda-\alpha}{1-\alpha}\right)\right]\eta(x,y) \\ &= y+\lambda\eta(x,y)=z. \end{aligned}$$

According to (4.1), we have

$$f(z)\leqslant b_1(x,z_2,\alpha)f(z_2)+b_2(x,z_2,\alpha)f(x).$$

Again, from (4.3) and the above inequality it follows that

$$f(z)<f(z_2). \tag{4.5}$$

Let

$$u=\frac{\lambda-\alpha}{(1-\alpha)\lambda}.$$

Since $0<\alpha<\lambda<1$, it is easy to show that $0<u<1$.

Thus

$$z+(1-u)\eta(y,z)=y+\lambda\eta(x,y)+(1-u)\eta(y,y+\lambda\eta(x,y))$$

$$= y + [\lambda - \lambda(1-u)]\eta(x, y)$$
$$= y + \lambda u\eta(x, y) = y + \frac{\lambda - \alpha}{1-\alpha}\eta(x, y) = z_2.$$

Since f is an explicitly B-preinvex function, from inequality (4.3) and the above equality, we obtain

$$f(z_2) < b_1(y,z,u)f(y) + b_2(y,z,u)f(z) < f(z),$$

which contradicts (4.5). □

Under the lower semicontinuity condition and Condition C, we prove that the explicit B-preinvexity implies the B-preinvexity.

Theorem 4.2. *Let K be a nonempty invex set in $\Re^n$ with respect to $\eta:\Re^n\times\Re^n \to \Re^n$, and $f:K\to\Re$ an explicitly B-preinvex function on K with respect to η, b_1, b_2. If f is a lower semicontinuous function and η satisfies Condition* C, *then f is a B-preinvex function on K with respect to the same η, b_1, b_2.*

Proof. Let $x, y\in K$. If $f(x)\neq f(y)$, then by the definition of an explicitly B-preinvex function with respect to η, b_1, b_2, we have

$$f(y + \lambda\eta(x, y)) < b_2(x, y, \lambda)f(x) + b_1(x, y, \lambda)f(y), \quad \forall\lambda\in(0, 1).$$

Now suppose that $f(x) = f(y)$. To show that f is a B-preinvex function with respect to η, b_1, b_2, we need to show that

$$f(y + \lambda\eta(x, y)) \leqslant f(x), \quad \forall\lambda\in(0, 1).$$

By contradiction, suppose that there exists an $\alpha\in(0, 1)$ such that

$$f(y + \alpha\eta(x, y)) > f(x). \tag{4.6}$$

Let $z_\alpha = y + \alpha\eta(x, y)$. Since f is lower semicontinuous, there exists β: $\alpha < \beta < 1$, such that

$$f(z_\beta) = f(y + \beta\eta(x, y)) > f(x) = f(y). \tag{4.7}$$

From Condition C, we have

$$z_\beta = z_\alpha + \frac{\beta - \alpha}{1-\alpha}\eta(x, z_\alpha).$$

Hence, from (4.6) and the explicit B-preinvexity of f, we have

$$f(z_\beta) < b_2\left(x, z_\alpha, \frac{\beta-\alpha}{1-\alpha}\right)f(x) + b_1\left(x, z_\alpha, \frac{\beta-\alpha}{1-\alpha}\right)f(z_\alpha) < f(z_\alpha). \tag{4.8}$$

On the other hand, from Condition C,

$$z_\alpha = z_\beta + \left(1 - \frac{\alpha}{\beta}\right)\eta(y, z_\beta).$$

Therefore, from (4.7) and explicit B-preinvexity of f, we have

$$f(z_\alpha) < b_1\left(z_\beta, y, \frac{\alpha}{\beta}\right) f(y) + b_2\left(z_\beta, y, \frac{\alpha}{\beta}\right) f(z_\beta) < f(z_\beta),$$

which contradicts (4.8). This completes the proof. □

5. Conclusions

In this paper, we have defined a new class of functions called explicitly B-preinvex functions and established that every local minimum of an explicitly B-preinvex function is also a global one. Various relationships between a B-preinvex function and an explicitly B-preinvex function were given under a lower semicontinuity and an intermediate-point B-preinvexity, respectively.

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